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On the master symmetries related to certain classes of integrable Hamiltonian systems

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Abstract. We present the complete classification of master symmetries related to a non-degenerate Hamiltonian system that is integrable in the Arnol'd–Liouville's sense. It is shown that a $\mathcal{C}$-integrable Hamiltonian system cannot have generators of degree greater than 2. Specific properties of the master symmetries classified in terms of the action–angle variables are investigated.

1. Introduction

The notion of a master symmetry, introduced in [1] by Fokas and Fuchssteiner, has been intensively studied in the framework of the theory of Hamiltonian dynamical systems [2, 3, 4, 5, 6, 7, 8, 9, 10]. These remarkable vector fields play an especially important role in the bi-Hamiltonian case, where the existence of a recursion operator provides a mechanism for generating an infinite hierarchy of master symmetries constituting a Virasoro-type Lie algebra [10]. Each member of such a hierarchy generates the corresponding hierarchies of Hamiltonian vector fields, their first integrals and Poisson (or symplectic) structures. This approach has been successfully applied to a number of well known systems of evolution equations [6, 7, 8, 9, 10].

Brouzet [11], studying non-degenerate integrable Hamiltonian systems, classified all symmetries of such systems in terms of the action–angle variables. One can extend this result to the master symmetries and investigate their properties using a similar approach. We note that ten Eikelder in [12], in a way considered an inverse problem for a class of Hamiltonian systems, showing that in a special system of coordinates the corresponding recursion operator, the Hamiltonian function and the symplectic form all have a special (diagonal) form. In this case the recursion operator generates an infinite sequence of non-trivial symmetries for the Hamiltonian system. It was also shown that non-degenerate Hamiltonian systems having action–angle coordinates enjoy all of those properties.

Consider an even-dimensional Poisson manifold (M^{2k}, P) defined by a non-degenerate Poisson bivector P^{ij}. A Hamiltonian system

$$\dot{x}^i = P^{i\alpha}\frac{\partial H}{\partial x^\alpha} \qquad i = 1, \ldots, 2k \tag{1}$$

is said to be *completely integrable in the Arnol'd–Liouville sense* if it has k functionally independent first integrals $\{F_1, F_2, \ldots, F_k\}$ in involution with respect to the Poisson bracket defined by P^{ij}:

$$\{F_i, F_j\}_P = P^{\alpha\beta}\frac{\partial F_i}{\partial x^\alpha}\frac{\partial F_j}{\partial x^\beta} = 0. \tag{2}$$

We use the Einstein summation convention. The Arnol'd–Liouville theorem [13] states that the map $\pi : M \to \mathbb{R}^k : m \to \{F_1(m), \ldots, F_k(m)\}$ produces the constant level surface $N_c = \{m \in M^{2k}, \pi(m) = c\}$ (we assume N_c to be connected), which is a submanifold of dimension k and there exists a contractible neighbourhood $V \in \mathbb{R}^k$ about $c \in \mathbb{R}^k$ such that $\pi^{-1}(V) = N_c \times V$. Then N_c is an invariant submanifold with respect to the Hamiltonian vector field X_H defining (1). The action–angle variables (I_i, φ_i), $i = 1, \ldots, k$ are obtained when N_c and V (being contractible) are diffeomorphic to a torus $\mathbb{T}^k$ or a toroidal cylinder $\mathbb{T}^m \times \mathbb{R}^{k-m}$ (if N_c is not compact) and an open ball $\mathbb{B}^k$, respectively. In this case the angle coordinates $\varphi_i, \ldots, \varphi_k$ run over a torus $\mathbb{T}^k$, $0 \leqslant \varphi_j \leqslant 2\pi$ (in the compact case) or over a cylinder $\mathbb{T}^m \times \mathbb{R}^{k-m}$ (if the submanifold N_c is not compact), while the action coordinates $I_1, \ldots, I_k$ are defined in an open ball $\mathbb{B}^k$. In these variables the completely integrable system (1) takes the form

$$\dot{I}_i = 0 \qquad \dot{\varphi}_i = \frac{\partial H}{\partial I_i} \qquad H = H(I_1, \ldots, I_k) \qquad i = 1, \ldots, k. \tag{3}$$

The symplectic structure $\omega := P^{-1}$ is canonical: $\omega = \sum_{i=1}^{k} \mathrm{d}I_i \wedge \mathrm{d}\varphi_i$ and the corresponding Hamiltonian vector field X_H becomes

$$X_H = \sum_{i=1}^{k} \frac{\partial H}{\partial I_i} \frac{\partial}{\partial \varphi_i}. \tag{4}$$

Then the system (1) is said to be *non-degenerate* if its Hessian has the maximum rank on a dense subset of $\mathbb{R}^k$, or

$$\det \left\| \frac{\partial^2 H(I)}{\partial I_i \partial I_j} \right\| \neq 0. \tag{5}$$

This implies that any first integral F of the Hamiltonian system (1) depends on the action variables only:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = 0 \quad \Rightarrow \quad F = F(I_1, \ldots, I_k) := F(I). \tag{6}$$

We note that a Hamiltonian system is called *$\mathcal{C}$-integrable* in a domain $\mathcal{O} \subset M$ iff it is integrable in the Arnol'd–Liouville sense, non-degenerate and in the domain $\mathcal{O}$ all invariant submanifolds of constant level of k involutive first integrals are compact (see [14]).

2. Master symmetries in the action–angle variables

The Brouzet lemma states that all symmetries of the non-degenerate integrable system (1) or the vector fields Y commuting with $X_H : [X_H, Y] = 0$ have the following form:

$$Y = \sum_{i=1}^{k} V^i(I) \frac{\partial}{\partial \varphi_i} \tag{7}$$

where (I_i, φ_i), $i = 1, \ldots, k$ are the corresponding action–angle coordinates. The representation (7) allows us to classify all master symmetries of (1), i.e. the vector fields Z satisfying

$$[X_H, [X_H, Z]] = 0 \tag{8}$$

provided $[X_H, Z] \neq 0$. Indeed, it follows from (8) that $\tilde{Y} := [X_H, Z]$ is a symmetry of X_H and so in the action–angle variables by the Brouzet lemma is given by

$$\tilde{Y} = \sum_{i=1}^{k} V^i(I) \frac{\partial}{\partial \varphi_i}. \tag{9}$$

Let us now assume that a master symmetry Z of the system (1) has the following general form:

$$Z(I,\varphi) = \sum_{i=1}^{k} U^i(I,\varphi)\frac{\partial}{\partial\varphi_i} + \sum_{i=1}^{k} W^i(I,\varphi)\frac{\partial}{\partial I_i} \qquad i = 1,\ldots,k. \tag{10}$$

Then, commuting $X_H = \sum_{i=1}^{k}(\partial H/\partial I_i)\,\partial/\partial\varphi_i$ with the vector field Z, we obtain

$$\tilde{Y} = \sum_{j=1}^{k} \frac{\partial H(I)}{\partial I_i}\frac{\partial U^j(I,\varphi)}{\partial\varphi_i}\frac{\partial}{\partial\varphi_j} + \sum_{j=1}^{k}\frac{\partial H(I)}{\partial I_i}\frac{\partial W^j(I,\varphi)}{\partial\varphi_i}\frac{\partial}{\partial I_j} - \sum_{j=1}^{k} W^i(I,\varphi)\frac{\partial^2 H(I)}{\partial I_i\partial I_j}\frac{\partial}{\partial\varphi_j}$$

$$= \sum_{j=1}^{k} V^j(I)\frac{\partial}{\partial\varphi_j}.$$

This leads to the following two conditions:

$$\frac{\partial W^i(I,\varphi)}{\partial\varphi_j} = 0 \tag{11}$$

$$\frac{\partial H(I)}{\partial I_i}\frac{\partial U^j(I,\varphi)}{\partial\varphi_i} - W^i(I,\varphi)\frac{\partial^2 H(I)}{\partial I_i\partial I_j} = V^j(I) \tag{12}$$

$$i, j = 1,\ldots,k.$$

Equation (11) implies that $W^i = W^i(I)$ for each $i = 1,\ldots,k$, for in this case $X_H(W^i) = 0$, i.e. W is a first integral of the system (1) and by (6) depends only on the action variables. From (12) we conclude that $\partial U^j(I,\varphi)/\partial\varphi_i$ for $i, j = 1,\ldots,k$ does not depend on the angle coordinates. Therefore U^j is an affine function of the variables φ_i; $i = 1,\ldots,k$. However, U is a global function and so is periodic in φ. Thus $U^j = U^j(I)$. This yields the general formula for a master symmetry of the Hamiltonian system (1) in the action–angle variables.

Lemma 2.1. Given a $\mathcal{C}$-integrable Hamiltonian system (3). Then, an arbitrary master symmetry $Z \in TM^{2k}$ of the corresponding Hamiltonian vector field X_H is given by the general formula

$$Z = \sum_{i=1}^{k} U^i(I)\frac{\partial}{\partial\varphi_i} + \sum_{i=1}^{k} W^i(I)\frac{\partial}{\partial I_i}. \tag{13}$$

The generic formula (13) enables us to verify many specific properties of master symmetries. For example, it is easy to see that the map $Z : \mathcal{F}M \to \mathcal{F}M$ defined by a master symmetry Z of the non-degenerate integrable Hamiltonian system (1) maps solutions to solutions. At the same time all master symmetries of (1) constitute a non-Abelian Lie algebra.

3. On the generator of degree *n*

The notion of a master symmetry admits a natural generalization [5]. We call a vector field Z *a generator of degree n* of a Hamiltonian vector field X_H if

$$L_{X_H}^n Z = 0$$

provided that $L_{X_H}^{n-1} Z \neq 0$. Here L_{X_H} denotes the usual Lie derivative along the vector field X_H : $L_{X_H} Z = [X_H, Z]$. Clearly, master symmetries and symmetries are generators of degrees 2 and 1, respectively. This notion was employed by Oevel and Falck [4] to investigate the problem of integrability of the Calogero–Moser system. We shall show that a $\mathcal{C}$-integrable non-degenerate Hamiltonian system cannot have generators of degree greater than 3. Indeed, assume that the system (1) is $\mathcal{C}$-integrable and has a generator of degree 3, i.e. there exists a vector field Z, such that

$$[X_H, [X_H, [X_H, Z]]] = 0$$

while $[X_H, [X_H, Z]] \neq 0$. Then $[X_H, Z]$ is a master symmetry of X_H, which, according to the generic formula (13), takes the following form:

$$[X_H, Z] = \sum_{i=1}^{k} U^i \frac{\partial}{\partial \varphi_i} + \sum_{i=1}^{k} W^i(I) \frac{\partial}{\partial I_i}. \tag{14}$$

On the other hand, for the vector field Z given by

$$Z = \sum_{i=1}^{k} Z_1^i(I, \varphi) \frac{\partial}{\partial \varphi_i} + \sum_{i=1}^{k} Z_2^i(I, \varphi) \frac{\partial}{\partial I_i} \tag{15}$$

we have

$$[X_H, Z] = \sum_{i=1}^{k} \frac{\partial H(I)}{\partial I_i} \frac{\partial Z_1^j(I, \varphi)}{\partial \varphi_i} \frac{\partial}{\partial \varphi_j} + \sum_{i=1}^{k} \frac{\partial H(I)}{\partial I_i} \frac{\partial Z_2^j(I, \varphi)}{\partial \varphi_i} \frac{\partial}{\partial I_j} - Z_2^i \frac{\partial^2 H(I)}{\partial I_i \partial I_j} \frac{\partial}{\partial \varphi_j}. \tag{16}$$

Comparing equations (14) and (16), we obtain

$$W^j(I) = \frac{\partial H(I)}{\partial I_i} \frac{\partial Z_2^j(I, \varphi)}{\partial \varphi_i} \tag{17}$$

and

$$U^j(I) = \frac{\partial H}{\partial I_i} \frac{\partial Z_1^j(I, \varphi)}{\partial \varphi_i} - Z_2^i \frac{\partial^2 H(I)}{\partial I_i \partial I_j} \tag{18}$$

for $i, j = 1, \ldots, k$. From (17) it follows that Z_2^i, $i = 1, \ldots, k$ are affine functions in the angle variables, while (18) suggests that Z_1^i, $i = 1, \ldots, k$ are also affine functions in $\varphi_1, \ldots, \varphi_k$ and Z_2 depends on the action variables only (we have used the condition of non-degeneracy (5)). Since $Z_1^i, i = 1, \ldots, k$ are globally defined on the compact torus, they also depend on the action variables only. Therefore, as follows from (13), Z is a master symmetry of X_H, which contradicts the initial assumption.

The same proof is applicable in the case of a generator of degree greater than 3. Therefore we arrive at the following conclusion.

Proposition 3.1. Any $\mathcal{C}$-integrable Hamiltonian system can only have generators of degree no greater than 2.

Compactness of the corresponding invariant submanifold is essential here; if this condition does not hold, the system can have generators of an arbitrary degree. For example, the above-mentioned Calogero–Moser system considered in [4] was shown to have generators of an arbitrary degree, since its constant level surface N_c was proved to be diffeomorphic to Liouville's *cylinder*, which was not compact.

4. The bi-Hamiltonian case

Now consider the bi-Hamiltonian case, namely that where the Hamiltonian vector field of the system (1) can be defined by two Poisson bivectors P_0 and P_1 with the following properties:
(i) The vector field X_H has two Hamiltonian representations:

$$X_H = P_0 dH_1 = P_1 dH_0 \tag{19}$$

where H_1 and H_0 are the corresponding Hamiltonians.
(ii) The linear operator $A := P_0 P_1^{-1}$ (assuming P_1 in non-degenerate) has a vanishing Nijenhuis tensor:

$$N_A(X, Y) = A^2[X, Y] + [AX, AY] - A([AX, Y] + [X, AY]) = 0 \tag{20}$$

for arbitrary vector fields $X, Y \in TM^{2k}$. In this case the operator A is called a *recursion operator* and has at least doubly degenerate eigenvalues [12], which are the first integrals of the vector field (19), in involution with respect to both Poisson bivectors P_0 and P_1. This leads to complete integrability in Arnol'd–Liouville's sense of the bi-Hamiltonian system (19) [13, 14, 15].

Now assume that equation (19) is a non-degenerate with respect to the Hamiltonian function H_1, and so can be defined in the action–angle variables (I_i, φ_i), because of its complete integrability. Then in these coordinates the operator A depends only on the action variables I_i. Indeed, the vector field $Y^i := \sum_{i=1}^{k} A^i_j \partial/\partial\varphi_j$ is a symmetry of $X_H = (\partial H_0/\partial I_i)\, \partial/\partial\varphi_i$, since $L_{X_H}(A) = 0$, and by the Leibniz rule

$$[X_H, Y] = L_{X_H}(Y) = A L_{X_H}\left(\frac{\partial}{\partial\varphi}\right) + L_{X_H}(A)\frac{\partial}{\partial\varphi} = 0.$$

Thus by the Brouzet lemma, the vector field Y has the representation (7) and so the recursion operator A depends only on the action variables.

Having a master symmetry Z_0 of X_H, one can construct an infinite hierarchy of master symmetries. This can be achieved by acting A^n on the initial master symmetry $Z_0 : Z_n := A^n Z_0,\ n = 1, \ldots$. It can easily be seen that if Z_0 is presented in the generic form (13), the map $\mathcal{M} : Z_0 \to Z_n := A^n Z_0$ is an automorphism up to the representation (13), since for each n the linear operator A^n depends on the action variables only. Thus all vector fields Z_n are given by (in the action–angle variables):

$$Z_n = \sum_{i=1}^{k} U^i_n(I)\frac{\partial}{\partial\varphi_i} + \sum_{i=1}^{k} W^i_n(I)\frac{\partial}{\partial I_i} \qquad i = 1, \ldots, k.$$

Therefore the hierarchy $\{Z_0, Z_1, \ldots, Z_n, \ldots\}$ consists of master symmetries of the system (19). Moreover, if the recursion operator A is non-degenerate, we can also extend this chain for negative integers n. The last conclusion was derived for a general system of coordinates [5] (although without the assumption of non-degeneracy) using lengthy calculations. Similar reasonings were employed in [12] to generate an infinite hierarchy of non-trivial symmetries. Finally, we conclude that the approach employed by Brouzet is proved to be quite effective for studying master symmetries.

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